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Tracking Control Design of Interval Type-2 Polynomial-Fuzzy-Model-Based Systems with Time-Varying Delay

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Abstract

In this paper, the tracking control design for the interval type-2 (IT2) polynomial-fuzzy-model-based (PFMB) control system subject to time-varying delay situation is investigated. The tracking control system is formed of the IT2 polynomial fuzzy model representing a nonlinear system with time-varying delay, the stable reference model and the IT2 polynomial fuzzy controller. The control objective is to design a proper IT2 polynomial fuzzy controller which is capable of driving the states of the polynomial fuzzy model to track those in the reference model and the tracking performance is evaluated and improved by the H_∞ performance index. Also, to handle the uncertainty in the membership functions, the property of IT2 fuzzy sets is utilized to enhance the fuzzy controller's robustness against uncertainty. In addition, considering the effect of time-varying delay, the Lyapunov-Krasovskii functional based approach is adopted to facilitate the delay-dependent stability

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analysis. Stability conditions depending on the time-varying delay characteristic with the consideration of H_∞ performance are obtained in terms of sum-of-squares (SOS). Furthermore, the information of the IT2 membership functions is employed in the stability analysis to relax the stability conditions, both membership-function-independent (MFI) and membership-function-dependent (MFD) approaches are presented to develop the stability conditions. Simulation examples are presented to verify the effectiveness of the proposed tracking control approach.

Keywords: Interval type-2 fuzzy sets, Polynomial-fuzzy-model-based (PFMB) control systems, Time-varying delay, Stability analysis, Sum-of-squares (SOS), Tracking control.

1. Introduction

Analysis of nonlinear control systems is generally challenging due to their complexity in nature. An effective way to represent the dynamics of the complex nonlinear control system is the Takagi-Sugeno (T-S) fuzzy-model-based (FMB) approach (Takagi and Sugeno, 1985; Sugeno and Kang, 1988), in which a family of local linear sub-systems are adopted and then fuzzily blended together smoothly through the membership functions to describe the global behavior of the nonlinear system. Thanks to its favorable model structure in support of control design and its rigorous mathematical foundation, the stability analysis and control synthesis of T-S FMB control systems can be conducted in a systematic way.

For the T-S FMB control systems, a popular approach to investigate the stability is based on Lyapunov stability theory. From the Lyapunov stability theory, if there is a common solution exists for all Lyapunov inequalities in

terms of linear matrix inequalities (LMIs), then the T-S FMB control system is guaranteed to be asymptotically stable (Wang et al., 1996; Tanaka and Wang, 2004). Given that the stability conditions of the T-S FMB control systems are in the form of LMIs, it can be solved numerically by convex programming techniques. There are fruitful research outcomes on the T-S FMB control problems, just to name a few, the works in (Wang et al., 1996; Tanaka et al., 1998; Kim and Lee, 2000; Tanaka and Wang, 2004; Liu and Zhang, 2003b,a; Teixeira et al., 2003; Fang et al., 2006; Sala and Ariño, 2007) are dedicated to relaxing the stability conditions of T-S FMB control systems; the works in (Nguang and Shi, 2003; Xu and Lam, 2005; Lin et al., 2005; Zhou et al., 2005) are of the H_∞ control design. Besides the stabilization problems of FMB control systems, the tracking control issues are frequently confronted in many control applications and the tracking control problems are generally considered to be more challenging than the stabilization problems. In the tracking control design, the controller is required to drive the states of the plant to track those of a stable reference model rather than just stabilize the plant (Tseng et al., 2001). Inspired by the success of applications of the FMB control approach, fuzzy tracking control technique was introduced in the work in (Tseng et al., 2001) and H_∞ performance index was considered to evaluate the tracking performance. There are many research results achieved on various tracking control problems such as output-feedback tracking control problems (Tseng, 2006; Lian and Liou, 2006; Lin et al., 2006; Mansouri et al., 2009; Lam and Li, 2013) and sampled-data output feedback is considered in the tracking control strategy (Lam and Seneviratne, 2009).

Recently, the T-S FMB control systems have been extended to the polynomial-fuzzy-model-based (PFMB) control systems by allowing poly-

nomial terms to exist in the model and the stability conditions of PFMB control system can be obtained in the forms of sum-of-squares (SOS)(Tanaka et al., 2009). The SOS-based stability conditions can be solved efficiently by using existing optimization techniques, for example, the SOSTOOLS(Prajna et al., 2004). The polynomial fuzzy model has more potential to represent the nonlinear dynamics than the T-S fuzzy model since when the order of the polynomial terms is 0, the polynomial fuzzy model will be reduced to a T-S fuzzy model. In the literature, the most popular type of membership functions used in the FMB/PFMB control systems is of type-1 fuzzy sets. It is well-known that the control strategies adopting type-1 fuzzy sets have been applied successfully to tackle the nonlinearities in control systems. However, the control strategies based on the type-1 fuzzy sets lack the ability to deal with the uncertainty directly. Uncertainties cannot be avoided under many situations (Mendel et al., 2006; Mendel, 2007), for example, the parameter uncertainties and different understanding of fuzzy rules from different people. To cope with the unavoidable uncertainties, the concept of footprint of uncertainty (FOU) is introduced along with the type-2 fuzzy sets (Mendel, 2007) to include the uncertainties into the type-2 membership functions. However, the the complexity of the control system is increased. To alleviate the complexity of the type-2 fuzzy sets based systems, the interval type-2 (IT2) fuzzy sets (Liang and Mendel, 2000; Mendel et al., 2006) are introduced as a compromise made between the type-1 and type-2 fuzzy sets. In IT2 fuzzy sets, the secondary grade of the membership is considered as a constant instead of a secondary function in the type-2 fuzzy sets. Relevant research of system control and stability analysis have been conducted recently based on the framework of IT2 FMB/PFMB control systems can be found in (Lam and Seneviratne, 2008; Juang and Hsu, 2009; Biglarbegian

et al., 2010; Jafarzadeh et al., 2011a,b; Lam et al., 2014; Li et al., 2016, 2015; Xiao et al., 2017; Song et al., 2017).

Time-delay appears commonly in various practical systems such as chemical processes, networked systems and communication systems (Zhao et al., 2009; Wu et al., 2011; Su et al., 2013; Yang et al., 2014), which is generally considered as the source of poor system performance and instability. Given that there are many complex nonlinear systems subject to time delay in practical situations and FMB control approaches are effective to represent the dynamics of nonlinear systems, it is natural to investigate nonlinear systems with time delay through the corresponding FMB control approaches (Cao and Frank, 2000). Therefore, the research of the FMB control system with time-delay is of great importance and researchers dedicated considerable effects to the problems of analysis and synthesis for time-delay FMB control systems. There are two main approaches to handle the time-delay problems in the literature, namely the delay-independent (Cao and Frank, 2000, 2001; Wang et al., 2004) and the delay-dependent (Guan and Chen, 2004; Chen et al., 2005; Zhou and Li, 2005; Chen et al., 2006; Wu, 2006; Wu and Li, 2007; Lam and Leung, 2007; Gao et al., 2009; Zhang and Xu, 2009; Zhao et al., 2009; Wu et al., 2011; Su et al., 2013; Yang et al., 2014; Wu et al., 2014) approaches. For the delay-independent approach, the stability conditions include no information of the delay, which means the stability conditions are guaranteed for arbitrary time delay. On the contrary, the delay-dependent approach contains the information of the delay, which is able to achieve less conservative results than the delay-independent approach, especially when the delay time is small. Within the delay-dependent approach, there are works on the constant time delay problems (Guan and Chen, 2004; Chen et al., 2005; Zhou and Li, 2005; Wu, 2006) and time-varying delay prob-

lems(Wu and Li, 2007; Gao et al., 2009; Zhang and Xu, 2009; Wu et al., 2011; Su et al., 2013; Yang et al., 2014; Wu et al., 2014). The advantage of the time-varying approach is that the constant time-delay case can be regarded as a special case of time-varying delay FMB control systems. It is also noticed that in the works in (Zhou et al., 2015; Li et al., 2017), the time delay issues based on IT2 fuzzy sets were investigated.

Having mentioned and reviewed the previous related works, to the authors' best knowledge, there is little literature on the tracking control design of IT2 PFMB control systems with time-varying delay, which is the main motivation for this paper. In this paper, the tracking control problem for the IT2 PFMB system is investigated under time-varying delay. To begin with the investigation, the IT2 polynomial fuzzy model is first built to represent the dynamics of the time-varying nonlinear plant. In the meantime, the uncertainty is included in the IT2 membership functions of the polynomial fuzzy model. An IT2 polynomial fuzzy controller is designed to drive the states of the IT2 polynomial fuzzy model to follow those of a reference model and the tracking performance is optimized according to the H_∞ performance. The optimization of the tracking performance in the analysis is formulated as the generalized eigenvalue minimization problem (GEVP). It should be noted that due to the uncertainty contained in the IT2 membership functions of the IT2 polynomial fuzzy model, the parallel distributed compensation (PDC)-based stability analysis in (Wang et al., 1996; Tanaka et al., 1998; Kim and Lee, 2000; Tanaka and Wang, 2004; Liu and Zhang, 2003b,a; Teixeira et al., 2003; Fang et al., 2006; Sala and Ariño, 2007) cannot be applied anymore since it requires that both the fuzzy model and the fuzzy controller share exactly the same premise membership functions. When the PDC approach is not applied, the analysis result can be

very conservative. To further relax the stability conditions, we suggest the membership-function-dependent (MFD) approach, in which the information of the membership functions can be included in the stability conditions in terms of SOS. On the contrary, the membership-function-independent (MFI) stability conditions do not take any information of membership functions into account, which means in the MFI approach, the stability conditions are guaranteed unnecessarily for all possible membership functions, which is the source of conservativeness. Therefore, the MFD stability conditions are more relaxed even though the computational demand is generally higher than the MFI ones (Lam, 2017). This conclusion will be verified by simulation examples in the paper.

This paper is organized as follows: In Section 2, the IT2 polynomial fuzzy model considering a time-varying delay, the IT2 polynomial fuzzy controller and the reference model are introduced. In Section 3, the stability of the IT2 PFMB tracking control system with time-varying delay is investigated and both the MFI and MFD SOS-based stability conditions are obtained. In Section 4, the simulation examples are given to show the effectiveness of the proposed approach. The conclusion is drawn in Section 5.

2. Preliminaries

In this section, the notations in the paper, the IT2 polynomial fuzzy model with time-varying delay, reference model and IT2 polynomial fuzzy controller are introduced.

Notations in the paper: the symbol “ \star ” denote the transposed elements in the symmetric positions of a matrix; The symbols “ $\mathbf{I}_{N,N}$ ” and “ $\mathbf{0}_{N,N}$ ” denote an identity matrix with appropriate dimensions (i.e., $\mathbf{I}_{N,N} \in \Re^{N \times N}$)

and an empty matrix with appropriate dimensions (i.e., $\mathbf{0}_{N,N} \in \mathbb{R}^{N \times N}$), respectively.

2.1. IT2 Polynomial Fuzzy Model with Time-Varying Delay

An IT2 polynomial fuzzy model considering time-varying delay with p rules is adopted to describe the dynamics of the nonlinear plant. The polynomial fuzzy model employed here is similar to those in the work in (Lam and Seneviratne, 2008; Lam et al., 2014; Xiao et al., 2017) where the main difference is that the time-varying delay is considered. The rules are in the following format, in which the antecedents are of IT2 fuzzy sets and the consequents are of polynomial systems:

Rule i : IF $f_1(\mathbf{x}(t))$ is \tilde{M}_1^i AND \dots AND $f_\Psi(\mathbf{x}(t))$ is \tilde{M}_Ψ^i THEN

$$\dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{A}_{di}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t-d(t))) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \quad (1)$$

$$\mathbf{x}(t) = \varphi(t), t \in [-\bar{d}, 0) \quad (2)$$

where \tilde{M}_α^i is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, \dots, \Psi$ and $i = 1, 2, \dots, p$; Ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times N}$, $\mathbf{A}_{di}(\mathbf{x}(t)) \in \mathbb{R}^{n \times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times m}$ are the polynomial system, polynomial delay and polynomial input matrices; $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \mathbb{R}^N$ is a vector of monomials in $\mathbf{x}(t)$, and $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector. $d(t) \in (0, \bar{d}]$ is the time-varying states delay and $\dot{d}(t) \leq \gamma$, in which γ is a constant and generally less than 1. $\varphi(t)$ is the initial condition. It is assumed that $\hat{\mathbf{x}} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$.

The firing strength of the i -th fuzzy rule is represented within the following interval sets:

$$\tilde{w}_i(\mathbf{x}(t)) \in [w_i^L(\mathbf{x}(t)), w_i^U(\mathbf{x}(t))], \quad i = 1, 2, \dots, p \quad (3)$$

where

$$w_i^L(\mathbf{x}(t)) = \prod_{\alpha=1}^{\Psi} \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))), \quad (4)$$

$$w_i^U(\mathbf{x}(t)) = \prod_{\alpha=1}^{\Psi} \bar{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \quad (5)$$

in which $0 \leq \bar{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \leq 1$ and $0 \leq \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \leq 1$ denote the upper and lower grades of membership governed by their upper and lower membership functions, respectively. From the definition of IT2 membership functions, we know that $0 \leq \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \leq \bar{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \leq 1$ and $0 \leq w_i^L(\mathbf{x}(t)) \leq w_i^U(\mathbf{x}(t)) \leq 1$ for all i .

$\tilde{w}_i(\mathbf{x}(t))$ is defined as follows:

$$\tilde{w}_i(\mathbf{x}(t)) = \underline{\lambda}_i(\mathbf{x}(t))w_i^L(\mathbf{x}(t)) + \bar{\lambda}_i(\mathbf{x}(t))w_i^U(\mathbf{x}(t)), \quad (6)$$

$$0 \leq \underline{\lambda}_i(\mathbf{x}(t)) \leq 1, \quad (7)$$

$$0 \leq \bar{\lambda}_i(\mathbf{x}(t)) \leq 1, \quad (8)$$

$$\underline{\lambda}_i(\mathbf{x}(t)) + \bar{\lambda}_i(\mathbf{x}(t)) = 1, \forall i \quad (9)$$

where $\underline{\lambda}_i(\mathbf{x}(t))$ and $\bar{\lambda}_i(\mathbf{x}(t))$ are nonlinear type reduction functions, which are related to the uncertainties.

Combining all the fuzzy rules together, the IT2 polynomial fuzzy model is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p \tilde{w}_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{A}_{di}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t-d(t))) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)) \quad (10)$$

where

$$\sum_{i=1}^p \tilde{w}_i(\mathbf{x}(t)) = 1, \quad \tilde{w}_i(\mathbf{x}(t)) \geq 0 \forall i. \quad (11)$$

2.2. Reference Model

The stable reference model adopted in the paper is defined as follows:

$$\dot{\mathbf{x}}_{\mathbf{r}}(t) = \mathbf{A}_{\mathbf{r}}\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t)) + \mathbf{B}_{\mathbf{r}}\mathbf{r}(t) \quad (12)$$

where $\mathbf{x}_{\mathbf{r}}(t) \in \Re^n$ is the system state vector of the reference model, which needs to be followed by the IT2 polynomial fuzzy model (10), $\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}_{\mathbf{r}}(t)$ as the entries, $\mathbf{A}_{\mathbf{r}} \in \Re^{n \times N}$ and $\mathbf{B}_{\mathbf{r}} \in \Re^{n \times m}$ are the constant system and input matrices, respectively, $\mathbf{r}(t) \in \Re^m$ is the reference input vector. It should be pointed out that the reference model is required to be stable but $\mathbf{A}_{\mathbf{r}}$ and $\mathbf{B}_{\mathbf{r}}$ do not necessarily have to be constant matrices, they can also be functions of the system states in the form of $\mathbf{A}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t))$ and $\mathbf{B}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t))$, respectively.

2.3. IT2 Polynomial Fuzzy Controller

An IT2 polynomial fuzzy controller with c rules is employed to actuate the states of the nonlinear plant represented by the IT2 polynomial fuzzy model (10) to track those in the reference model.

The rules and consequents of the IT2 polynomial fuzzy controller is defined as follows:

$$\begin{aligned} \text{Rule } j : & \text{ IF } g_1(\mathbf{x}(t)) \text{ is } \tilde{N}_1^j \text{ AND } \dots \text{ AND } g_{\Omega}(\mathbf{x}(t)) \text{ is } \tilde{N}_{\Omega}^j \\ & \text{ THEN } \mathbf{u}(t) = \mathbf{F}_j(\mathbf{x}(t))\hat{\mathbf{e}}(t) + \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t)) \end{aligned} \quad (13)$$

where \tilde{N}_{β}^j is an IT2 fuzzy term of rule j corresponding to function $g_{\beta}(\mathbf{x}(t))$, $\beta = 1, 2, \dots, \Omega$ and $j = 1, 2, \dots, c$; Ω is a positive integer; $\mathbf{F}_j(\mathbf{x}(t))$ and $\mathbf{G}_j(\mathbf{x}(t)) \in \Re^{m \times N}$, $j = 1, 2, \dots, c$, are the polynomial feedback gains to be designed. $\hat{\mathbf{e}}(t) = \hat{\mathbf{x}}(\mathbf{x}(t)) - \hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t))$ represents the difference between the states in the fuzzy model and the reference model.

Inspired by (Lam et al., 2014), $\tilde{m}_j(\mathbf{x}(t)) \geq 0$ is defined as follows:

$$\tilde{m}_j(\mathbf{x}(t)) = \frac{\underline{\kappa}_j(\mathbf{x}(t))m_j^L(\mathbf{x}(t)) + \bar{\kappa}_j(\mathbf{x}(t))m_j^U(\mathbf{x}(t))}{\sum_{k=1}^c \left(\underline{\kappa}_k(\mathbf{x}(t))m_k^L(\mathbf{x}(t)) + \bar{\kappa}_k(\mathbf{x}(t))m_k^U(\mathbf{x}(t)) \right)} \quad (14)$$

where $m_j^L(\mathbf{x}(t)) = \prod_{l=1}^{\Omega} \underline{\mu}_{\tilde{N}_l^j}(g_l(\mathbf{x}(t)))$, $m_j^U(\mathbf{x}(t)) = \prod_{l=1}^{\Omega} \bar{\mu}_{\tilde{N}_l^j}(g_l(\mathbf{x}(t)))$, in which $0 \leq \bar{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \leq 1$ and $0 \leq \underline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \leq 1$ denote the upper and lower grades of membership governed by the lower and upper membership functions, respectively. By the definition of IT2 membership functions, the property $0 \leq \underline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \leq \bar{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \leq 1$ holds and further leads to the $0 \leq m_j^L(\mathbf{x}(t)) \leq m_j^U(\mathbf{x}(t)) \leq 1$ valid for all j . $0 \leq \underline{\kappa}_j(\mathbf{x}(t)) \leq 1$, $0 \leq \bar{\kappa}_j(\mathbf{x}(t)) \leq 1$, $\underline{\kappa}_j(\mathbf{x}(t)) + \bar{\kappa}_j(\mathbf{x}(t)) = 1$, $\forall j$; $\underline{\kappa}_j(\mathbf{x}(t))$ and $\bar{\kappa}_j(\mathbf{x}(t))$ are nonlinear functions to be determined.

After fuzzy blending all the rules together, the IT2 polynomial fuzzy controller is described by

$$\mathbf{u}(t) = \sum_{i=1}^c \tilde{m}_j(\mathbf{x}(t)) \left(\mathbf{F}_j(\mathbf{x}(t))\hat{\mathbf{e}}(t) + \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t)) \right) \quad (15)$$

where

$$\sum_{i=1}^c \tilde{m}_j(\mathbf{x}(t)) = 1, \tilde{m}_j(\mathbf{x}(t)) \geq 0, \forall j. \quad (16)$$

3. Main Result

The control objective is to design a proper IT2 polynomial fuzzy controller to drive the states of the fuzzy model to follow those of the stable reference model closely. In the following analysis, the H_∞ index will be adopted to describe the level of the tracking error and the performance of the tracking control system will be improved subject to the H_∞ performance. Properly designed IT2 polynomial fuzzy controller is able to attenuate the

difference between the states of the IT2 polynomial fuzzy model and reference system $\mathbf{e}(t)$ as small as possible, which means the improved H_∞ control performance can be achieved.

For brevity, in the following context, the time t associated with the variables is omitted in the situation which does not cause ambiguity, e.g., $\hat{\mathbf{e}}(t)$, $\mathbf{x}(t)$, $\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t))$, $\mathbf{r}(t)$ and $\hat{\mathbf{x}}(\mathbf{x}(t))$ are denoted as $\hat{\mathbf{e}}$, \mathbf{x} , $\hat{\mathbf{x}}_{\mathbf{r}}$, \mathbf{r} and $\hat{\mathbf{x}}$, respectively. Also $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ are denoted as w_i and m_j in the context without ambiguity.

3.1. Stability Conditions with H_∞ Performance

Connecting the IT2 polynomial fuzzy model with time-varying delay (10) and the IT2 polynomial fuzzy controller (15), we get the closed-loop dynamic equation as follows:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j (\mathbf{A}_i(\mathbf{x}) \hat{\mathbf{x}} + \mathbf{A}_{di}(\mathbf{x}) \hat{\mathbf{x}}(t - d(t)) + \mathbf{B}_i(\mathbf{x}) \mathbf{u}) \\ &= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \left(\mathbf{A}_i(\mathbf{x}) \hat{\mathbf{x}} + \mathbf{A}_{di}(\mathbf{x}) \hat{\mathbf{x}}(t - d(t)) + \mathbf{B}_i(\mathbf{x}) (\mathbf{F}_j(\mathbf{x}) \mathbf{e} + \mathbf{G}_j(\mathbf{x}) \mathbf{x}_{\mathbf{r}}) \right)\end{aligned}\quad (17)$$

in which $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ and $\hat{\mathbf{x}}(\mathbf{x}) = [\hat{x}_1(\mathbf{x}), \hat{x}_2(\mathbf{x}), \dots, \hat{x}_N(\mathbf{x})]$.

To adopt $\dot{\hat{\mathbf{x}}}$ in the analysis, let us consider the relationship between $\dot{\hat{\mathbf{x}}}$ and $\dot{\mathbf{x}}$, which can be linked together by $\mathbf{T}(\mathbf{x})$ as follows:

$$\dot{\hat{\mathbf{x}}} = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x}) \dot{\mathbf{x}}, \quad (18)$$

in which $\mathbf{T}(\mathbf{x}) \in \Re^{N \times n}$ with its $\alpha\beta$ -th element $T_{\alpha\beta}(\mathbf{x})$ defined as

$$T_{\alpha\beta}(\mathbf{x}) = \frac{\partial \hat{x}_\alpha(\mathbf{x})}{\partial x_\beta}, \alpha = 1, 2, \dots, N; \beta = 1, 2, \dots, n. \quad (19)$$

From (17) and (18), we have the dynamics of $\dot{\hat{\mathbf{x}}}$ as follows:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) \hat{\mathbf{x}} + \tilde{\mathbf{A}}_{di}(\mathbf{x}) \hat{\mathbf{x}}(t-d(t)) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{u}) \\ &= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \left(\tilde{\mathbf{A}}_i(\mathbf{x}) \hat{\mathbf{x}} + \tilde{\mathbf{A}}_{di}(\mathbf{x}) \hat{\mathbf{x}}(t-d(t)) + \tilde{\mathbf{B}}_i(\mathbf{x}) (\mathbf{F}_j(\mathbf{x}) \mathbf{e} + \mathbf{G}_j(\mathbf{x}) \mathbf{x}_r) \right)\end{aligned}\quad (20)$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \mathbf{A}_i(\mathbf{x})$, $\tilde{\mathbf{A}}_{di}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \mathbf{A}_{di}(\mathbf{x})$ and $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \mathbf{B}_i(\mathbf{x})$.

Meanwhile, $\dot{\hat{\mathbf{x}}}(\mathbf{x}_r)$ will also be adopted in the analysis, let us consider the reference model (12), in which $\mathbf{x}_r = [x_{r_1}, x_{r_2}, \dots, x_{r_n}]^T$ and $\hat{\mathbf{x}}(\mathbf{x}_r) = [\hat{x}_{r_1}(\mathbf{x}_r), \hat{x}_{r_2}(\mathbf{x}_r), \dots, \hat{x}_{r_N}(\mathbf{x}_r)]^T$, we have the polynomial dynamic model for the reference model:

$$\dot{\hat{\mathbf{x}}}(\mathbf{x}_r) = \frac{\partial \hat{\mathbf{x}}_r(\mathbf{x}_r)}{\partial \mathbf{x}_r} \frac{d\mathbf{x}_r}{dt} = \mathbf{H}(\mathbf{x}_r) \dot{\mathbf{x}}_r = \tilde{\mathbf{A}}_r(\mathbf{x}_r) \hat{\mathbf{x}}_r + \tilde{\mathbf{B}}_r(\mathbf{x}_r) \mathbf{r} \quad (21)$$

where $\tilde{\mathbf{A}}_r(\mathbf{x}_r) = \mathbf{H}(\mathbf{x}_r) \mathbf{A}_r$, $\tilde{\mathbf{B}}_r(\mathbf{x}_r) = \mathbf{H}(\mathbf{x}_r) \mathbf{B}_r$ and $\mathbf{H}(\mathbf{x}_r) \in \mathbb{R}^{N \times n}$ with its $\alpha\beta$ -th element is defined as

$$H_{\alpha\beta}(\mathbf{x}_r) = \frac{\partial \hat{x}_{r_\alpha}(\mathbf{x}_r)}{\partial x_{r_\beta}}, \alpha = 1, 2, \dots, N; \beta = 1, 2, \dots, n. \quad (22)$$

In order to apply the Lyapunov approach to develop the stability conditions, we define $0 < \mathbf{X} = \mathbf{X}^T \in \mathbb{R}^{N \times N}$. The dynamics of $\hat{\mathbf{e}}$ is obtained

as

$$\begin{aligned}
\dot{\mathbf{e}} &= \dot{\hat{\mathbf{x}}} - \dot{\hat{\mathbf{x}}}_{\mathbf{r}} = \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{F}_j(\mathbf{x})) \hat{\mathbf{e}} \\
&+ \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_{di}(\mathbf{x}) \hat{\mathbf{x}}(t-d(t)) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x}) \hat{\mathbf{x}}_{\mathbf{r}}) \\
&+ \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})) \hat{\mathbf{x}}_{\mathbf{r}} - \tilde{\mathbf{B}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) \mathbf{r} \\
&= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{F}_j(\mathbf{x})) \hat{\mathbf{e}} + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \tilde{\mathbf{A}}_{di}(\mathbf{x}) \hat{\mathbf{x}}_{\mathbf{r}}(t-d(t)) \\
&+ \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \tilde{\mathbf{A}}_{di}(\mathbf{x}) \hat{\mathbf{e}}(t-d(t)) + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j ((\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})) \hat{\mathbf{x}}_{\mathbf{r}} \\
&+ \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x}) \hat{\mathbf{x}}_{\mathbf{r}}) - \tilde{\mathbf{B}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) \mathbf{r} \\
&= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \Phi_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \mathbf{z}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}, \mathbf{x}(t-d(t)), \mathbf{x}_{\mathbf{r}}(t-d(t)), \mathbf{r}) \tag{23}
\end{aligned}$$

where $\Phi_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) = [\Phi_{ij}^{(1)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \quad \Phi_{ij}^{(2)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \quad \Phi_{ij}^{(3)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \quad \Phi_{ij}^{(4)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \quad \Phi_{ij}^{(5)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})]$
 $= [\tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X} + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{M}_j(\mathbf{x}) \quad \tilde{\mathbf{A}}_{di}(\mathbf{x})\mathbf{X} \quad (\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}))\mathbf{X} + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) \quad \tilde{\mathbf{A}}_{di}(\mathbf{x})\mathbf{X}$
 $- \tilde{\mathbf{B}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})], \mathbf{z}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}, \mathbf{x}(t-d(t)), \mathbf{x}_{\mathbf{r}}(t-d(t)), \mathbf{r}) = [\hat{\mathbf{e}}^T \mathbf{X}^{-1} \quad \hat{\mathbf{e}}^T(t-d(t)) \mathbf{X}^{-1}$
 $\hat{\mathbf{x}}_{\mathbf{r}}^T \mathbf{X}^{-1} \quad \hat{\mathbf{x}}_{\mathbf{r}}^T(t-d(t)) \mathbf{X}^{-1} \quad \mathbf{r}^T]^T$, in which $\mathbf{M}_j(\mathbf{x}) = \mathbf{F}_j(\mathbf{x})\mathbf{X}$ and $\mathbf{N}_j(\mathbf{x}) = \mathbf{G}_j(\mathbf{x})\mathbf{X}$.

Remark 1. To lighten the notation burden, $\mathbf{z}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}, \mathbf{x}(t-d(t)), \mathbf{x}_{\mathbf{r}}(t-d(t)), \mathbf{r})$ is replaced by \mathbf{z} in the following analysis.

Theorem 1. Consider an IT2 PFMB tracking control system, which is formed by a nonlinear plant represented by the polynomial fuzzy model with time-varying delay (10) and the IT2 polynomial fuzzy controller (15) connected in a closed loop. Given a time-varying delay function $d(t)$, in which $0 < d(t) \leq \bar{d}$ and $\dot{d}(t) \leq \gamma$, its system states are driven to follow those of the

reference model (12) where the tracking error is subject to an H_∞ performance characterized by the scalars $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_3 > 0$, if there exist matrices $\mathbf{X} = \mathbf{X}^T \in \mathbb{R}^{N \times N}$, $\mathbf{Z} = \mathbf{Z}^T \in \mathbb{R}^{N \times N}$, $\mathbf{S} \in \mathbb{R}^{N \times N}$, $\mathbf{Q} = \mathbf{Q}^T \in \mathbb{R}^{N \times N}$, $\mathbf{M}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$ and $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$, $j = 1, 2, \dots, c$ such that the following GEVP is feasible:

$$\begin{aligned}
& \min \sigma_1 + \sigma_2 + \sigma_3 \text{ subject to} \\
& \sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0; \\
& \nu_1^T (\mathbf{X} - \varepsilon_1 \mathbf{I}) \nu_1 \text{ is SOS;} \\
& \nu_1^T (\mathbf{Z} - \varepsilon_2 \mathbf{I}) \nu_1 \text{ is SOS;} \\
& \nu_1^T (\mathbf{Q} - \varepsilon_3 \mathbf{I}) \nu_1 \text{ is SOS;} \\
& \nu_2^T \left(\begin{bmatrix} \mathbf{Z} & \mathbf{S} \\ \star & \mathbf{Z} \end{bmatrix} - \varepsilon_4 \mathbf{I} \right) \nu_2 \text{ is SOS;} \\
& -\nu_3^T (\Psi_{ij}(\mathbf{x}, \mathbf{x}_r) + \varepsilon_5(\mathbf{x}, \mathbf{x}_r) \mathbf{I}) \nu_3 \text{ is SOS,} \\
& i = 1, 2, \dots, p; j = 1, 2, \dots, c
\end{aligned} \tag{24}$$

where $\nu_1 \in \mathbb{R}^N$, $\nu_2 \in \mathbb{R}^{2N}$ and $\nu_3 \in \mathbb{R}^{6N+m}$ are arbitrary vectors independent of \mathbf{x} and \mathbf{x}_r ; $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$ and $\varepsilon_5(\mathbf{x}, \mathbf{x}_r) > 0$ are predefined scalars and polynomial, respectively,

$$\Psi_{ij}(\mathbf{x}, \mathbf{x}_r) = \begin{bmatrix} \Xi_{ij}(\mathbf{x}, \mathbf{x}_r) & \star \\ \bar{d}\Phi_{ij}(\mathbf{x}, \mathbf{x}_r)\Lambda & -\bar{d}(2\zeta\mathbf{X} - \zeta^2\mathbf{Z}) \end{bmatrix}, \tag{25}$$

ζ is a scalar chosen by the user, and

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \\ \Lambda_5 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N,N} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{N,N} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{N,N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{N,N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{m,m} \end{bmatrix}, \quad (26)$$

$$\Xi_{ij}(\mathbf{x}, \mathbf{x}_r) = \begin{bmatrix} \Pi_{ij}^{(11)} & \Pi_{ij}^{(12)} & \Pi_{ij}^{(13)} & \Pi_{ij}^{(14)} & \Pi_{ij}^{(15)} & \Pi_{ij}^{(16)} \\ \star & \Pi_{ij}^{(22)} & \Pi_{ij}^{(23)} & \Pi_{ij}^{(24)} & \Pi_{ij}^{(25)} & \Pi_{ij}^{(26)} \\ \star & \star & \Pi_{ij}^{(33)} & \Pi_{ij}^{(34)} & \Pi_{ij}^{(35)} & \Pi_{ij}^{(36)} \\ \star & \star & \star & \Pi_{ij}^{(44)} & \Pi_{ij}^{(45)} & \Pi_{ij}^{(46)} \\ \star & \star & \star & \star & \Pi_{ij}^{(55)} & \Pi_{ij}^{(56)} \\ \star & \star & \star & \star & \star & \Pi_{ij}^{(66)} \end{bmatrix}, \quad (27)$$

$$\begin{aligned} \Pi_{ij}^{(11)} &= \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X} + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{M}_j(\mathbf{x}) + \mathbf{X}\tilde{\mathbf{A}}_i^T(\mathbf{x}) + \mathbf{M}_j^T(\mathbf{x})\tilde{\mathbf{B}}_i^T(\mathbf{x}) + \mathbf{Q} - \frac{1}{d}\mathbf{Z} + \mathbf{I}_{N,N}; \\ \Pi_{ij}^{(12)} &= \tilde{\mathbf{A}}_{di}(\mathbf{x})\mathbf{X} + \frac{1}{d}(\mathbf{Z} - \mathbf{S}); \Pi_{ij}^{(13)} = \mathbf{S}; \Pi_{ij}^{(14)} = (\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_r(\mathbf{x}_r))\mathbf{X} + \\ &\tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}); \Pi_{ij}^{(15)} = \tilde{\mathbf{A}}_{di}(\mathbf{x})\mathbf{X}; \Pi_{ij}^{(16)} = -\tilde{\mathbf{B}}_r(\mathbf{x}_r); \Pi_{ij}^{(22)} = \mathbf{I}_{N,N} - (1 - \gamma)\mathbf{Q} - \\ &\frac{1}{d}(2\mathbf{Z} - \mathbf{S}^T - \mathbf{S}); \Pi_{ij}^{(23)} = \frac{1}{d}(-\mathbf{S} + \mathbf{Z}); \Pi_{ij}^{(24)} = \mathbf{0}_{N,N}; \Pi_{ij}^{(25)} = \mathbf{0}_{N,N}; \Pi_{ij}^{(26)} = \\ &\mathbf{0}_{N,m}; \Pi_{ij}^{(33)} = \mathbf{I}_{N,N} - \frac{1}{d}\mathbf{Z}; \Pi_{ij}^{(34)} = \mathbf{0}_{N,N}; \Pi_{ij}^{(35)} = \mathbf{0}_{N,N}; \Pi_{ij}^{(36)} = \mathbf{0}_{N,m}; \\ \Pi_{ij}^{(44)} &= -\sigma_1\mathbf{I}_{N,N}; \Pi_{ij}^{(45)} = \mathbf{0}_{N,N}; \Pi_{ij}^{(46)} = \mathbf{0}_{N,m}; \Pi_{ij}^{(55)} = -\sigma_2\mathbf{I}_{N,N}; \Pi_{ij}^{(56)} = \\ &\mathbf{0}_{N,m}; \Pi_{ij}^{(66)} = -\sigma_3\mathbf{I}_{m,m}. \end{aligned}$$

Proof. A Lyapunov-Krasovskii functional is defined as follows to include the time-varying function $d(t)$ in the analysis and develop the delay-time

dependent stability conditions:

$$\begin{cases} V(t) = V_1(t) + V_2(t) + V_3(t), \\ V_1(t) = \hat{\mathbf{e}}^T(t) \mathbf{X}^{-1} \hat{\mathbf{e}}(t) \\ V_2(t) = \int_{-\bar{d}}^0 \int_{t+\sigma}^t \dot{\hat{\mathbf{e}}}^T(s) \mathbf{R} \dot{\hat{\mathbf{e}}}(s) ds d\sigma \\ V_3(t) = \int_{t-d(t)}^t \hat{\mathbf{e}}^T(s) \mathbf{U} \hat{\mathbf{e}}(s) ds \end{cases} \quad (28)$$

where $\mathbf{R} = \mathbf{R}^T > 0 \in \mathbb{R}^{N \times N}$ and $\mathbf{U} = \mathbf{U}^T > 0 \in \mathbb{R}^{N \times N}$. It can be seen that $V(t)$ is a positive function. The three parts of $\dot{V}(t)$ are obtained as follows:

$$\dot{V}_1(t) = 2\hat{\mathbf{e}}^T(t) \mathbf{X}^{-1} \dot{\hat{\mathbf{e}}}(t), \quad (29)$$

$$\dot{V}_2(t) = \bar{d} \dot{\hat{\mathbf{e}}}^T(t) \mathbf{R} \dot{\hat{\mathbf{e}}}(t) - \int_{t-\bar{d}}^t \dot{\hat{\mathbf{e}}}^T(s) \mathbf{R} \dot{\hat{\mathbf{e}}}(s) ds, \quad (30)$$

$$\dot{V}_3(t) = \hat{\mathbf{e}}^T(t) \mathbf{U} \dot{\hat{\mathbf{e}}}(t) - (1 - \dot{d}(t)) \hat{\mathbf{e}}^T(t - d(t)) \mathbf{U} \hat{\mathbf{e}}(t - d(t)) \quad (31)$$

$$\leq \hat{\mathbf{e}}^T(t) \mathbf{U} \dot{\hat{\mathbf{e}}}(t) - (1 - \gamma) \hat{\mathbf{e}}^T(t - d(t)) \mathbf{U} \hat{\mathbf{e}}(t - d(t)). \quad (32)$$

Before we proceed further, the following lemma is introduced to deal with the integration component in $\dot{V}_2(t)$:

Lemma 1. *From (Park et al., 2011), for any matrix $\begin{bmatrix} \mathbf{R} & \mathbf{W} \\ \star & \mathbf{R} \end{bmatrix} \geq 0$,*

scalar $d(t) \in (0, \bar{d}]$, vector function $\dot{\hat{\mathbf{x}}} : [-\bar{d}, 0] \rightarrow \mathbb{R}^n$ such that the concerned integration in the following inequality is well defined, then

$$-\bar{d} \int_{t-\bar{d}}^t \dot{\hat{\mathbf{x}}}^T(s) \mathbf{R} \dot{\hat{\mathbf{x}}}(s) ds \leq \mathbf{v}(t)^T \Omega \mathbf{v}(t) \quad (33)$$

where

$$\mathbf{v}(t) = [\hat{\mathbf{x}}^T(t) \quad \hat{\mathbf{x}}^T(t-d(t)) \quad \hat{\mathbf{x}}^T(t-\bar{d})]^T \quad (34)$$

$$\Omega = \begin{bmatrix} -\mathbf{R} & \mathbf{R} - \mathbf{W} & \mathbf{W} \\ \star & -2\mathbf{R} + \mathbf{W}^T + \mathbf{W} & -\mathbf{W} + \mathbf{R} \\ \star & \star & -\mathbf{R} \end{bmatrix}. \quad (35)$$

This lemma suggests an upper bound for the integral term $-\int_{t-\bar{d}}^t \dot{\mathbf{e}}^T(s) \mathbf{R} \dot{\mathbf{e}}(s) ds$ in $\dot{V}_2(t)$ by introducing the newly defined $\mathbf{v}(t)$ and the matrices \mathbf{R} , \mathbf{W} and Ω , which will further make the stability conditions in favorable form of SOS.

Adopting Lemma 1 with small modification on the variables, $\dot{V}_2(t)$ can be rewritten as follows:

$$\begin{aligned} \dot{V}_2(t) &= \bar{d} \dot{\mathbf{e}}^T(t) \mathbf{R} \dot{\mathbf{e}}(t) - \int_{t-\bar{d}}^t \dot{\mathbf{e}}^T(s) \mathbf{R} \dot{\mathbf{e}}(s) ds \\ &\leq \bar{d} \dot{\mathbf{e}}^T(t) \mathbf{R} \dot{\mathbf{e}}(t) + \frac{1}{\bar{d}} \eta^T(t) \Omega \eta(t) \end{aligned} \quad (36)$$

where

$$\eta(t) = [\hat{\mathbf{e}}^T(t) \mathbf{X}^{-1} \quad \hat{\mathbf{e}}^T(t-d(t)) \mathbf{X}^{-1} \quad \hat{\mathbf{e}}^T(t-\bar{d}) \mathbf{X}^{-1}]^T, \quad (37)$$

$$\Omega = \begin{bmatrix} -\mathbf{Z} & \mathbf{Z} - \mathbf{S} & \mathbf{S} \\ \star & -2\mathbf{Z} + \mathbf{S}^T + \mathbf{S} & -\mathbf{S} + \mathbf{Z} \\ \star & \star & -\mathbf{Z} \end{bmatrix}, \quad (38)$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{S} \\ \star & \mathbf{Z} \end{bmatrix} \geq 0 \quad (39)$$

where $\mathbf{Z} = \mathbf{X} \mathbf{R} \mathbf{X}$, $\mathbf{S} = \mathbf{X} \mathbf{W} \mathbf{X}$ and $\mathbf{Q} = \mathbf{X} \mathbf{U} \mathbf{X}$.

Remark 2. It is worth mentioning that through adopting Lemma 1 in the analysis, the upper bound of the integral term $-\int_{t-\bar{d}}^t \dot{\mathbf{e}}^T(s) \mathbf{R} \dot{\mathbf{e}}(s) ds$ in $\dot{V}_2(t)$

can be obtained by the matrices \mathbf{Z} , \mathbf{S} and the vector $\eta(t)$. In the vector $\eta(t)$, the time-varying delay function $d(t)$ and the upper bound of $d(t)$ denoted by \bar{d} are included in the analysis, which makes the stability conditions delay-dependent.

In the following analysis, the newly defined augment vector $\xi = [\xi_1^T \ \xi_2^T \ \xi_3^T \ \xi_4^T \ \xi_5^T \ \xi_6^T]^T = [\dot{\mathbf{e}}^T \mathbf{X}^{-1} \ \dot{\mathbf{e}}^T(t-d(t))\mathbf{X}^{-1} \ \dot{\mathbf{e}}^T(t-\bar{d})\mathbf{X}^{-1} \ \hat{\mathbf{x}}_{\mathbf{r}}^T \mathbf{X}^{-1} \ \hat{\mathbf{x}}_{\mathbf{r}}^T(t-d(t))\mathbf{X}^{-1} \ \mathbf{r}^T]^T$ is adopted to facilitate the stability analysis.

Associate ξ with scalars σ_1 , σ_2 and σ_3 , $\dot{V}(t)$ can be rewritten as

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \xi^T \Xi_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \xi - \xi_1^T \xi_1 - \xi_2^T \xi_2 - \xi_3^T \xi_3 \\ & + \sigma_1 \xi_4^T \xi_4 + \sigma_2 \xi_5^T \xi_5 + \sigma_3 \xi_6^T \xi_6 + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \bar{d} \dot{\mathbf{e}}^T(t) \mathbf{R} \dot{\mathbf{e}}(t) \end{aligned} \quad (40)$$

where $\Xi_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})$ is given in (27).

In order to transform the last term in (40) into matrix form and further relate it with $\Xi_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})$, let us recall the definitions of $\dot{\mathbf{e}}$, $\Phi_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})$ and \mathbf{z} in (23), then the following equation can be obtained:

$$\sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \bar{d} \dot{\mathbf{e}}^T(t) \mathbf{R} \dot{\mathbf{e}}(t) = \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \bar{d} \mathbf{z}^T \Phi_{ij}^T(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \mathbf{R} \Phi_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \mathbf{z}. \quad (41)$$

To further process on the right hand side of (41), Λ defined in (26) is adopted here and it obtains that $\mathbf{z} = \Lambda \xi$ from the definition of \mathbf{z} in (23). The relationship between \mathbf{z} and ξ can be utilized in (41) to facilitate the

stability analysis, considering the summation of first and last terms in (40):

$$\begin{aligned}
& \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \xi^T \Xi_{ij}(\mathbf{x}, \mathbf{x}_r) \xi + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \bar{d} \mathbf{z}^T \Phi_{ij}^T \mathbf{R} \Phi_{ij} \mathbf{z} \\
&= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \xi^T \Xi_{ij}(\mathbf{x}, \mathbf{x}_r) \xi + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \bar{d} \xi^T \Lambda^T \Phi_{ij}^T \mathbf{R} \Phi_{ij} \Lambda \xi \\
&= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \xi^T (\Xi_{ij}(\mathbf{x}, \mathbf{x}_r) + \bar{d} \Lambda^T \Phi_{ij}^T \mathbf{R} \Phi_{ij} \Lambda) \xi. \tag{42}
\end{aligned}$$

From Schur complement lemma and using the property that $\mathbf{R}^{-1} \geq (2\zeta \mathbf{X} - \zeta^2 \mathbf{Z})(\text{Lam, 2012})$ where ζ is a scalar to be chosen, (42) is guaranteed to be negative definite if

$$\sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j \Psi_{ij}(\mathbf{x}, \mathbf{x}_r) < 0 \tag{43}$$

where $\Psi_{ij}(\mathbf{x}, \mathbf{x}_r)$ is defined in (25).

It can be found that the inequality (43) can be achieved by requiring that $\Psi_{ij}(\mathbf{x}, \mathbf{x}_r) < 0$, which leads to $\dot{V}(t) \leq -\xi_1^T \xi_1 - \xi_2^T \xi_2 - \xi_3^T \xi_3 + \sigma_1 \xi_4^T \xi_4 + \sigma_2 \xi_5^T \xi_5 + \sigma_3 \xi_6^T \xi_6$.

Considering the termination time of control t_f (Tseng et al., 2001; Lam and Li, 2013), the H_∞ performance of the tracking control system can be guaranteed as

$$\frac{\int_0^{t_f} (\xi_1^T \xi_1 + \xi_2^T \xi_2 + \xi_3^T \xi_3) - V(0)}{\int_0^{t_f} \sigma_1 \xi_4^T \xi_4 + \sigma_2 \xi_5^T \xi_5 + \sigma_3 \xi_6^T \xi_6} \leq 1. \tag{44}$$

□

Remark 3. In the H_∞ performance index (44), ξ_1 , ξ_2 and ξ_3 are referred to the tracking error of the control system $\hat{\mathbf{e}}(t)$, $\hat{\mathbf{e}}(t - d(t))$ and $\hat{\mathbf{e}}(t - \bar{d})$ while ξ_4 , ξ_5 and ξ_6 are related to the states of reference system and its input \mathbf{r} . Smaller positive values of σ_1 , σ_2 and σ_3 will help reduce the value of $\xi_1^T \xi_1 + \xi_2^T \xi_2 + \xi_3^T \xi_3$ indicating an improved H_∞ performance.

3.2. MFD Relaxed Stability Conditions with H_∞ Performance

After obtaining the stability conditions in Theorem 1, the next step is to further relax the stability conditions by utilizing the information of the membership functions. By applying the sub-domains approach (Xiao et al., 2017), the whole operation domain Φ of the premise variable \mathbf{x} is partitioned into L connected sub-domains Φ_l , $l = 1, 2, \dots, L$ such that $\Phi = \bigcup_{l=1}^L \Phi_l$. Considering the membership functions in sub-domains, a new function $\hat{h}_{ijl}(\mathbf{x})$ is defined to approximate the membership functions $\tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x})$ for \mathbf{x} in sub-domain Φ_l . Associate with the approximation functions $\hat{h}_{ijl}(\mathbf{x})$, the constant approximation error $\underline{\delta}_{ijl}$ and $\bar{\delta}_{ijl}$ are defined as follows:

$$\underline{\delta}_{ijl} \leq \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x}) - \hat{h}_{ijl}(\mathbf{x}) \leq \bar{\delta}_{ijl}, \forall i, j, l, \mathbf{x} \in \Phi_l. \quad (45)$$

The function $\hat{h}_{ijl}(\mathbf{x})$ can be a constant, linear function, or polynomial function of \mathbf{x} , as long as it can be handled in the solution finding processing. Generally speaking, the more complex the function is, the more precise approximation performance can be achieved under the same number of sub-domains, which means more relaxed stability conditions are expected. Also, increasing the number of sub-domains contributes to the reduction the conservativeness in the stability conditions. However, complex approximation functions and a large number of sub-domains will raise the computational burden on the numerical simulations, therefore the trade-off between the relaxation and the computational burden has to be made when applying the stability conditions.

Remark 4. *It is worth mentioning that it is the approximation function $\hat{h}_{ijl}(\mathbf{x})$ used in the stability analysis but not the original membership function $\tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x})$ due to the complexity of the original membership functions.*

Since the original membership function is nonlinear in general then it cannot be handled by numerical software like *SOSTOOLS*, therefore $\hat{h}_{ijl}(\mathbf{x})$ is defined as a simpler form such that it can be processed by *SOSTOOLS* even though the approximation error $\underline{\delta}_{ijl}$ and $\bar{\delta}_{ijl}$ will be generated in the analysis. Through employing $\hat{h}_{ijl}(\mathbf{x})$, $\underline{\delta}_{ijl}$ and $\bar{\delta}_{ijl}$ in the stability analysis, the information of the membership functions will be included in the analysis and further contribute to reducing the conservativeness lying in the stability conditions obtained in Theorem 1.

Remark 5. It should be pointed out that when the information of the membership functions into the analysis, different premise membership functions of the controller will lead to different control performance since the different premise membership function case might give different bounds of performance index (H_∞ index in this paper). In addition, the design of the controller can be more flexible when different membership functions from the fuzzy model can be applied to the design of a fuzzy controller. Generally, fewer fuzzy rules for the controller save the cost of the controller. Having said that, if the control objective is emphasized more on the performance, the membership functions of the fuzzy controller can be chosen in a way to achieve better performance.

In addition, to take the state information into account, the state boundaries of Φ_l will also be included in the analysis. Let us denote the lower and upper boundaries of Φ_l as $\underline{\mathbf{x}}_l$ and $\bar{\mathbf{x}}_l$, respectively. Then the following inequality will be included in the analysis:

$$(\mathbf{x} - \underline{\mathbf{x}}_l)^T \mathbf{D}(\bar{\mathbf{x}}_l - \mathbf{x}) \mathbf{L}_l(\mathbf{x}) \geq 0, \forall l \quad (46)$$

where $\mathbf{D} = \text{diag}\{d_1, d_2, \dots, d_n\} \in \Re^{n \times n}$ is a diagonal matrix, in which the values of d_1 to d_n can only be either 1 or 0. When the value of d_i is 1, it

means that the information of the x_i , which is in the sub-domain divided by $\underline{\mathbf{x}}_l$ and $\bar{\mathbf{x}}_l$, is considered during the analysis, otherwise, it is not considered. $\mathbf{L}_l(\mathbf{x}) = \mathbf{L}_l^T(\mathbf{x}) > 0 \in \Re^{(6N+m) \times (6N+m)}$ is a polynomial slack matrix to be determined.

To reduce the computational burden, the constant form \hat{h}_{ijl} is adopted for $\hat{h}_{ijl}(\mathbf{x})$ and with the condition $\underline{\delta}_{ijl} \leq \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x}) - \hat{h}_{ijl} \leq \bar{\delta}_{ijl}$, we can further define that $\underline{h}_{ijl} = \hat{h}_{ijl} + \underline{\delta}_{ijl}$, which is considered as the lower bound of $\tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x})$ in the sub-domain Φ_l , to facilitate the stability analysis.

Consider (45) and (46), and define the matrix $\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r) = \mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r)^T \in \Re^{(6N+m) \times (6N+m)} > 0$ which satisfies $\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r) \geq \mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r)$ for all i, j and l . Let us consider (43) in sub-domain Φ_l :

$$\begin{aligned}
& \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x})\mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r) \\
&= \sum_{i=1}^p \sum_{j=1}^c \hat{h}_{ijl}\mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r) + \sum_{i=1}^p \sum_{j=1}^c (\tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x}) - \hat{h}_{ijl} - \underline{\delta}_{ijl} + \underline{\delta}_{ijl})\mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r) \\
&= \sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ijl} + \underline{\delta}_{ijl})\mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r) + \sum_{i=1}^p \sum_{j=1}^c (\tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x}) - \hat{h}_{ijl} - \underline{\delta}_{ijl})\mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r) \\
&\leq \sum_{i=1}^p \sum_{j=1}^c \left((\hat{h}_{ijl} + \underline{\delta}_{ijl})\mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r) + (\bar{\delta}_{ijl} - \underline{\delta}_{ijl})\mathbf{Y}_{ij}(\mathbf{x}, \mathbf{x}_r) \right) \\
&= \sum_{i=1}^p \sum_{j=1}^c \left(\underline{h}_{ijl}\mathbf{\Psi}_{ij}(\mathbf{x}, \mathbf{x}_r) + (\bar{\delta}_{ijl} - \underline{\delta}_{ijl})\mathbf{Y}_{ij}(\mathbf{x}, \mathbf{x}_r) \right), \forall \mathbf{x} \in \Phi_l. \tag{47}
\end{aligned}$$

Denoting $\delta_{ijl} \equiv \bar{\delta}_{ijl} - \underline{\delta}_{ijl}$ and associating it with the boundaries infor-

mation in (46) into (47), we have

$$\begin{aligned}
& \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(\mathbf{x}) \tilde{m}_j(\mathbf{x}) \Psi_{ij}(\mathbf{x}, \mathbf{x}_r) \\
& \leq \sum_{i=1}^p \sum_{j=1}^c \underline{h}_{ijl} \Psi_{ij}(\mathbf{x}, \mathbf{x}_r) + \sum_{i=1}^p \sum_{j=1}^c \delta_{ijl} \mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r) \\
& + (\mathbf{x} - \underline{\mathbf{x}}_l)^T \mathbf{D}(\bar{\mathbf{x}}_l - \mathbf{x}) \mathbf{L}_l(\mathbf{x}), \forall \mathbf{x} \in \Phi_l.
\end{aligned} \tag{48}$$

When the right hand side of (48) is negative definite for every subdomain Φ_l , the system stability can be guaranteed and the IT2 polynomial fuzzy controller defined in (15) is able to drive the states of the fuzzy model (10) to follow those of the reference model (12) subject to the prescribed H_∞ performance in (44). The analysis results are summarized in the following theorem.

Theorem 2. *Considering the IT2 PFMB tracking control system with time-varying delay, which is formed by a nonlinear plant represented by the IT2 polynomial fuzzy model with time-varying delay (10) and the IT2 polynomial fuzzy controller (15) connected in a closed loop. Given a time-varying delay function $d(t)$, in which $0 < d(t) \leq \bar{d}$ and $\dot{d}(t) \leq \gamma$, its system states are driven to follow those of the stable reference model (12) where the tracking error is subject to an H_∞ performance characterized by the scalars $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 > 0$, if there exist matrices $\mathbf{L}_l(\mathbf{x}) = \mathbf{L}_l(\mathbf{x})^T \in \Re^{(6N+m) \times (6N+m)}$, $l = 1, 2, \dots, L$, $\mathbf{X} = \mathbf{X}^T \in \Re^{N \times N}$, $\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r) = \mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r)^T \in \Re^{(6N+m) \times (6N+m)}$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, c$, $l = 1, 2, \dots, L$, $\mathbf{Z} = \mathbf{Z}^T \in \Re^{N \times N}$, $\mathbf{S} \in \Re^{N \times N}$, $\mathbf{Q} = \mathbf{Q}^T \in \Re^{N \times N}$, $\mathbf{M}_j(\mathbf{x}) \in \Re^{m \times N}$ and $\mathbf{N}_j(\mathbf{x}) \in \Re^{m \times N}$ for $j = 1, 2, \dots, c$ such that the following GEVP is*

feasible:

$$\begin{aligned}
& \min \sigma_1 + \sigma_2 + \sigma_3 \text{ subject to} \\
& \sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0; \\
& \rho^T (\mathbf{L}_l(\mathbf{x}) - \varepsilon_1(\mathbf{x})\mathbf{I})\rho \text{ is SOS, } \forall l; \\
& \nu_1^T (\mathbf{X} - \varepsilon_2\mathbf{I})\nu_1 \text{ is SOS;} \\
& \nu_1^T (\mathbf{Z} - \varepsilon_2\mathbf{I})\nu_1 \text{ is SOS;} \\
& \nu_1^T (\mathbf{Q} - \varepsilon_4\mathbf{I})\nu_1 \text{ is SOS;} \\
& \nu_2^T \left(\begin{bmatrix} \mathbf{Z} & \mathbf{S} \\ \star & \mathbf{Z} \end{bmatrix} - \varepsilon_5\mathbf{I} \right) \nu_2 \text{ is SOS;} \\
& \rho^T (\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r) - \varepsilon_6(\mathbf{x}, \mathbf{x}_r)\mathbf{I})\rho \text{ is SOS, } \forall i, j, l; \\
& \rho^T (\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r) - \Psi_{ij}(\mathbf{x}, \mathbf{x}_r) - \varepsilon_7(\mathbf{x}, \mathbf{x}_r)\mathbf{I})\rho \text{ is SOS, } \forall i, j, l; \\
& -\rho^T \left(\sum_{i=1}^p \sum_{j=1}^c (\underline{h}_{ijl} \Psi_{ij}(\mathbf{x}, \mathbf{x}_r) + \delta_{ijl} \mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x}_r)) \right. \\
& \quad \left. + (\mathbf{x} - \underline{\mathbf{x}}_l)^T \mathbf{D}(\bar{\mathbf{x}}_l - \mathbf{x})\mathbf{L}_l(\mathbf{x}) + \varepsilon_8(\mathbf{x}, \mathbf{x}_r)\mathbf{I} \right) \rho \text{ is SOS, } \forall l
\end{aligned}$$

where $\nu_1 \in \mathbb{R}^N$ and $\nu_2 \in \mathbb{R}^{2N}$ are arbitrary vectors independent of \mathbf{x} and \mathbf{x}_r , $\rho \in \mathbb{R}^{6N+m}$ is an arbitrary vector independent of \mathbf{x} and \mathbf{x}_r , \underline{h}_{ijl} and δ_{ijl} are the predefined constants determined by the upper and lower bounds of the IT2 membership functions; $\Psi_{ij}(\mathbf{x}, \mathbf{x}_r)$ is defined in (25), $\mathbf{D} = \text{diag}\{d_1, d_2, \dots, d_n\} \in \mathbb{R}^{n \times n}$ is a predefined diagonal matrix, $\varepsilon_1(\mathbf{x}) > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, $\varepsilon_5 > 0$, $\varepsilon_6(\mathbf{x}, \mathbf{x}_r) > 0$, $\varepsilon_7(\mathbf{x}, \mathbf{x}_r) > 0$, $\varepsilon_8(\mathbf{x}, \mathbf{x}_r) > 0$ are predefined scalars or polynomials, $\underline{\mathbf{x}}_l$ and $\bar{\mathbf{x}}_l$ are the predefined lower and upper bounds of the system states \mathbf{x} , respectively, in the l -th operating sub-domain.

4. Simulation Examples

4.1. Numerical Example

In this section, a simulation example is given to verify the tracking control strategy. To demonstrate the effectiveness of the proposed approach, we design an IT2 PFMB tracking control system with time-varying delay equipped with the IT2 polynomial fuzzy controller to track the states of the reference model. Let us consider a three-rule polynomial fuzzy model with $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{x}(t) = [x_1(t) \ x_2(t)]^T$,

$$\mathbf{A}_1(x_1(t)) = \begin{bmatrix} 0.59 - 0.12x_1(t) & -7.29 - 1.82x_1(t) \\ 0.01 & -2.85 \end{bmatrix},$$

$$\mathbf{A}_2(x_1(t)) = \begin{bmatrix} 0.02 + 2.25x_1(t) & -4.64 + 0.72x_1(t) \\ 0.35 & -8.56 \end{bmatrix},$$

$$\mathbf{A}_3(x_1(t)) = \begin{bmatrix} 0.73 + 0.45x_1(t) & 8.45 + 2.13x_1(t) \\ 0.26 & -15.43 \end{bmatrix},$$

$$\mathbf{A}_{d1} = \begin{bmatrix} -0.05 & 0 \\ 0 & -0.05 \end{bmatrix}, \mathbf{A}_{d2} = \begin{bmatrix} -0.07 & 0 \\ 0 & -0.05 \end{bmatrix},$$

$$\mathbf{A}_{d3} = \begin{bmatrix} -0.05 & 0 \\ 0 & -0.07 \end{bmatrix},$$

$$\mathbf{B}_1(x_1(t)) = \begin{bmatrix} 1 + 1.35x_1(t) + 2.33x_1(t)^2 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_2(x_1(t)) = \begin{bmatrix} 8 - 0.62x_1(t) \\ 0 \end{bmatrix},$$

$$\mathbf{B}_3(x_1(t)) = \begin{bmatrix} 4 - 0.73x_1(t) + 3.35x_1(t)^2 \\ 0.8 \end{bmatrix}.$$

The lower and upper membership functions are chosen as $\underline{w}_1(x_1(t)) = 1 - 1/(1 + e^{(-x_1(t)-3.3)})$, $\underline{w}_3(x_1(t)) = 1/(1 + e^{(-x_1(t)+3.3)})$, $\bar{w}_2(x_1(t)) = 1 - \underline{w}_1(x_1(t)) - \underline{w}_3(x_1(t))$; $\bar{w}_1(x_1(t)) = 1 - 1/(1 + e^{(-x_1(t)-2.7)})$, $\bar{w}_3(x_1(t)) = 1/(1 + e^{(-x_1(t)+2.7)})$, $\underline{w}_2(x_1(t)) = 1 - \bar{w}_1(x_1(t)) - \bar{w}_3(x_1(t))$.

A two-rule IT2 polynomial fuzzy controller is employed to realize the tracking control purpose where the lower and upper membership functions are chosen as follows:

$$\underline{m}_1(x_1(t)) = \begin{cases} 1 & \text{for } x_1(t) < -5.15 \\ \frac{-x_1(t)+4.85}{10} & \text{for } -5.15 \leq x_1(t) \leq 4.85 \\ 0 & \text{for } x_1(t) > 4.85 \end{cases}, \quad (49)$$

$$\bar{m}_1(x_1(t)) = \begin{cases} 1 & \text{for } x_1(t) < -4.85 \\ \frac{-x_1(t)+5.15}{10} & \text{for } -4.85 \leq x_1(t) \leq 5.15 \\ 0 & \text{for } x_1(t) > 5.15 \end{cases}, \quad (50)$$

$\bar{m}_2(x_1(t)) = 1 - \underline{m}_1(x_1(t))$, $\underline{m}_2(x_1(t)) = 1 - \bar{m}_1(x_1(t))$, and $\tilde{m}_2(x_1(t)) = 1 - \tilde{m}_1(x_1(t))$.

The reference model is chosen as $\mathbf{A}_r = \begin{bmatrix} -1 & -1 \\ 0.25 & -10.5 \end{bmatrix}$, $\mathbf{B}_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $r(t) = 5\sin(0.4t)$.

It is supposed that the system is working under $x_1(t) \in [-10 \ 10]$. By dividing the operating domain $x_1(t)$ into 15 uniform sub-domains (i.e., $L =$

15), we have $\underline{x}_{1_l} = -\frac{34}{3} + \frac{4}{3}l$ and $\bar{x}_{1_l} = -10 + \frac{4}{3}l$, $l = 1, 2, \dots, 15$, which are the lower and upper bounds of the l -th sub-domains of the operating domain, respectively. It is chosen that $\mathbf{D} = \text{diag}\{1, 0\}$.

By applying the stability conditions in Theorem 2, it is chosen that $\varepsilon_1(\mathbf{x}) = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6(\mathbf{x}, \mathbf{x}_r) = \varepsilon_7(\mathbf{x}, \mathbf{x}_r) = \varepsilon_8(\mathbf{x}, \mathbf{x}_r) = 0.001$. $\mathbf{F}_j(x_1(t))$ and $\mathbf{G}_j(x_1(t))$, $j = 1, 2$ are polynomials with monomials in x_1 of degree 2; The upper bound of the delay time \bar{d} is chosen as 0.05 and the time-varying delay function is defined as $d(t) = 0.4\bar{d}\sin(t) + 0.6\bar{d}$ and γ , which is the upper bound of $\dot{d}(t)$, is 0.02.

After the stability conditions in the form of SOS have been solved, the polynomial feedback gains are obtained as $\mathbf{F}_1 = [-0.0008x_1^2 - 0.1412x_1 - 2.2063 \quad -0.0027x_1^2 - 0.0039x_1 + 0.1278]$, $\mathbf{F}_2 = [-0.0141x_1^2 + 0.3017x_1 - 2.2637 \quad 0.0043x_1^2 - 0.0481x_1 - 0.0248]$, $\mathbf{G}_1 = [0.000019x_1^2 - 0.0113x_1 - 0.1101 \quad -0.0011x_1^2 - 0.0187x_1 - 0.1377]$, $\mathbf{G}_2 = [-0.0007x_1^2 + 0.0318x_1 - 0.2283 \quad 0.0034x_1^2 - 0.0107x_1 - 0.3032]$ and $\mathbf{X} = \begin{bmatrix} 1.4875 & 0.0600 \\ 0.0600 & 0.9217 \end{bmatrix}$. The minimum values of σ_1 , σ_2 and σ_3 are obtained as 0.4548, 0.3502 and 16.5725, respectively.

In the time-response simulation, the type-reduction functions are chosen that $\underline{\lambda}_1(x_1(t)) = (\sin(5x_1(t)) + 1)/2$, $\bar{\lambda}_1(x_1(t)) = 1 - \underline{\lambda}_1(x_1(t))$, $\underline{\lambda}_3(x_1(t)) = (\cos(5x_1(t)) + 1)/2$, $\bar{\lambda}_3(x_1(t)) = 1 - \underline{\lambda}_3(x_1(t))$, which act as the uncertainty of the nonlinear plant embedded in the IT2 membership functions to obtain $\tilde{w}_1(x_1(t))$ and $\tilde{w}_3(x_1(t))$. Since $\tilde{w}_2(x_1(t)) = 1 - \tilde{w}_1(x_1(t)) - \tilde{w}_3(x_1(t))$ by the definition of $\tilde{w}_i(x_1(t))$, there is no need to obtain the explicit form of $\underline{\lambda}_2(x_1(t))$ and $\bar{\lambda}_2(x_1(t))$ once $\tilde{w}_1(x_1(t))$ and $\tilde{w}_3(x_1(t))$ are defined.

On the other hand, the type reduction functions for the controller are chosen as $\underline{\kappa}_j(x_1(t)) = \bar{\kappa}_j(x_1(t)) = 0.5$, $j = 1, 2$. By applying the IT2

polynomial fuzzy controller for tracking control with the initial conditions $\mathbf{x}(0) = [0 \ 0]$ and $\mathbf{x}_r(0) = [0.5 \ 0]$, the simulation results of state response and control signal are shown in Figs. 1 to 3, which demonstrate that the tracking errors are sufficiently small.

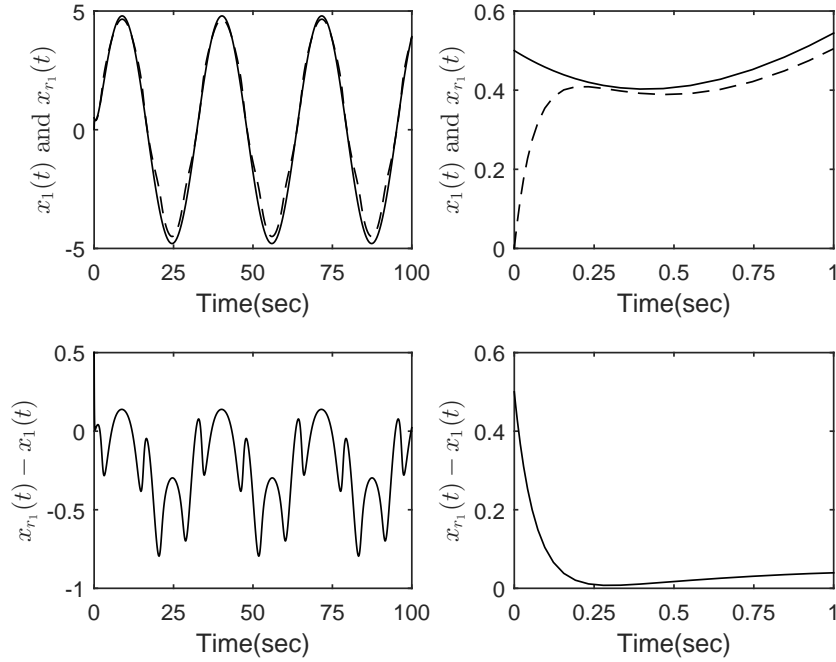


Figure 1: Tracking control performance for $x_1(t)$ under 15 sub-domains approach in the example, $\sigma_1 = 0.4548$, $\sigma_2 = 0.3502$ and $\sigma_3 = 16.5725$. Top left panel: responses of $x_1(t)$ (Dash curve) and $x_{r1}(t)$ (Solid curve) from 0 to 100 seconds. Top right panel: responses of $x_1(t)$ (Dash curve) and $x_{r1}(t)$ (Solid curve) from 0 to 1 second. Bottom left panel: response of $x_{r1}(t) - x_1(t)$ from 0 to 100 seconds. Bottom right panel: response of $x_{r1}(t) - x_1(t)$ from 0 to 1 second.

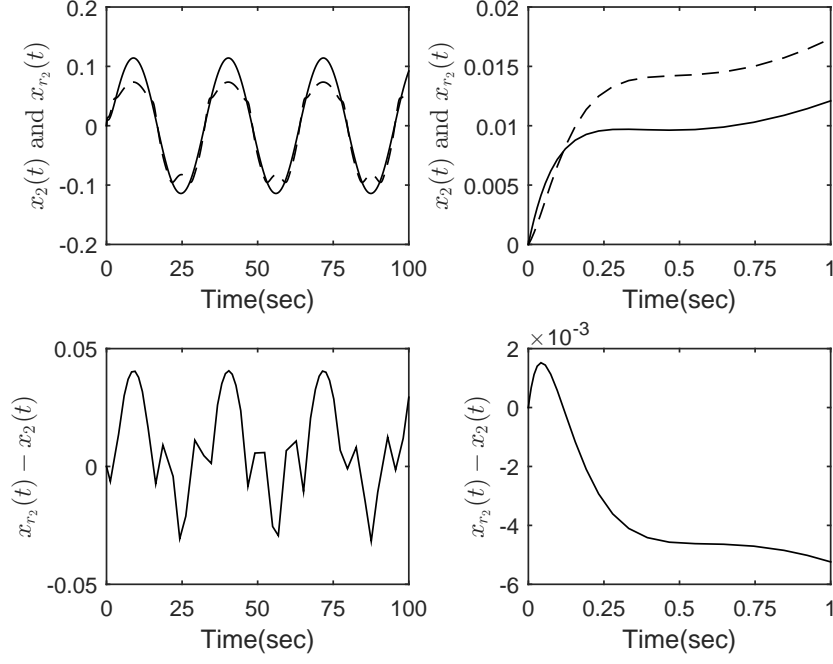


Figure 2: Tracking control performance for $x_2(t)$ under 15 sub-domains approach in the example, $\sigma_1 = 0.4548$ $\sigma_2 = 0.3502$ and $\sigma_3 = 16.5725$. Top left panel: responses of $x_2(t)$ (Dash curve) and $x_{r_2}(t)$ (Solid curve) from 0 to 100 seconds. Top right panel: responses of $x_2(t)$ (Dash curve) and $x_{r_2}(t)$ (Solid curve) from 0 to 1 second. Bottom left panel: response of $x_{r_2}(t) - x_2(t)$ from 0 to 100 seconds. Bottom right panel: response of $x_{r_2}(t) - x_2(t)$ from 0 to 1 second.

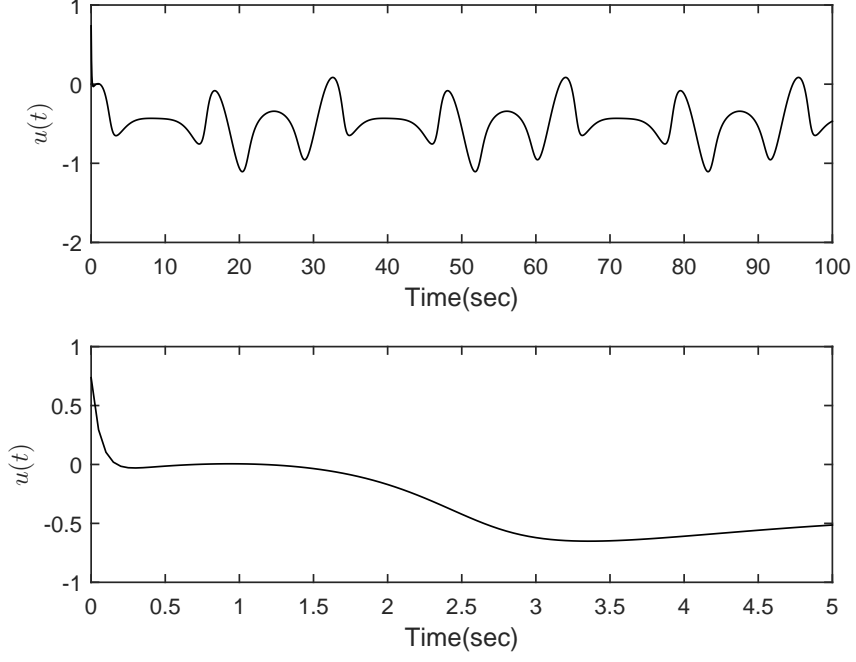


Figure 3: The control signal $u(t)$ under 15 sub-domains approach in the example. Top panel: $u(t)$ from 0 to 100 seconds. Bottom panel: $u(t)$ from 0 to 5 seconds.

Remark 6. When the feedback gains $\mathbf{F}_j(x_1(t))$ and $\mathbf{G}_j(x_1(t))$, $j = 1, 2$, are of degree 0 in $x_1(t)$, with all of other settings being the same, no feasible solution can be found.

Remark 7. To compare the MFD and MFI approaches, Theorem 1 is applied with the same IT2 polynomial fuzzy model, reference model and all other parameters parameters. No feasible solution is found, which demonstrates that introducing the information of IT2 membership functions helps reduce the conservativeness of stability conditions.

4.2. Inverted Pendulum

In this section, the inverted pendulum example is provided to verify the control strategy we propose, the dynamics of the inverted pendulum can be viewed in the following equation (Lam and Seneviratne, 2008):

$$\ddot{\theta}(t) = \frac{g \sin(\theta(t)) - am_p S \dot{\theta}(t)^2 \sin(2\theta(t))/2 - a \cos(\theta(t))u(t)}{4S/3 - am_p S \cos^2(\theta(t))} \quad (51)$$

where $\theta(t)$ is the angular displacement of the inverted pendulum, $g = 9.8 \text{ m/s}^2$, $m_p \in [m_{p_{\min}} \ m_{p_{\max}}] = [1 \ 2] \text{ kg}$ is the mass of the pendulum, $M_c \in [M_{c_{\min}} \ M_{c_{\max}}] = [18 \ 20] \text{ kg}$ is the mass of the cart, $a = \frac{1}{m_p + M_c}$, $2S = 1\text{m}$ is the length of the pendulum, and $u(t)$ is the control input applied on the cart. m_p and M_c are treated as the parameters subject to uncertainties.

Let us define the state variables as: $\hat{\mathbf{x}}(t) = \mathbf{x}(t) = [x_1(t) \ x_2(t)]^T = [\theta(t) \ \dot{\theta}(t)]^T$, in which $x_1(t) \in [-\frac{5\pi}{12} \ \frac{5\pi}{12}]$, $x_2(t) \in [-4 \ 4]$.

In order to write the dynamic equation of the inverted pendulum into state-space equation, let us introduce to nonlinear functions

$$f_1(\mathbf{x}(t)) = \frac{g - am_p S x_2(t)^2 \cos(x_1(t))}{4S/3 - am_p S \cos^2(x_1(t))} \left(\frac{\sin(x_1(t))}{x_1(t)} \right) \quad (52)$$

and

$$f_2(\mathbf{x}(t)) = \frac{-a \cos(x_1(t))}{4S/3 - am_p S \cos^2(x_1(t))}. \quad (53)$$

By Adopting $f_1(\mathbf{x}(t))$ and $f_2(\mathbf{x}(t))$, we can reform the nonlinear dynamic equation into IT2 polynomial fuzzy models. The following 4 rules are used to describe the inverted pendulum with time-varying delay (Li et al., 2018):

Rule i : IF $f_1(\mathbf{x}(t))$ is \tilde{M}_1^i AND $f_2(\mathbf{x}(t))$ is \tilde{M}_2^i

THEN $\dot{\mathbf{x}}(t) = (1 - \delta_d)\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \delta_d\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t - d(t))) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)$, (54)

where

$$\begin{aligned}\mathbf{A}_1(\mathbf{x}(t)) = \mathbf{A}_2(\mathbf{x}(t)) &= \begin{bmatrix} 0 & 1 \\ f_{1_{\min}}(\mathbf{x}(t)) & 0 \end{bmatrix}, \\ \mathbf{A}_3(\mathbf{x}(t)) = \mathbf{A}_4(\mathbf{x}(t)) &= \begin{bmatrix} 0 & 1 \\ f_{1_{\max}}(\mathbf{x}(t)) & 0 \end{bmatrix}, \\ \mathbf{B}_1(\mathbf{x}(t)) = \mathbf{B}_3(\mathbf{x}(t)) &= \begin{bmatrix} 0 \\ f_{2_{\min}}(\mathbf{x}(t)) \end{bmatrix}, \\ \mathbf{B}_2(\mathbf{x}(t)) = \mathbf{B}_4(\mathbf{x}(t)) &= \begin{bmatrix} 0 \\ f_{2_{\max}}(\mathbf{x}(t)) \end{bmatrix},\end{aligned}$$

$\delta_d = 0.05$ is the delay coefficient, the upper bound of the delay time \bar{d} is chosen as 0.05 and the time-varying delay function is defined as $d(t) = 0.4\bar{d}\sin(t) + 0.6\bar{d}$ and γ , which is the upper bound of $\dot{d}(t)$, is 0.02.

Considering all the fuzzy rules together, the IT2 polynomial fuzzy model contains time-varying delay is obtained as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^4 \tilde{w}_i \left((1 - \delta_d) \mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) + \delta_d \mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t - d(t))) + \mathbf{B}_i(\mathbf{x}(t)) \mathbf{u}(t) \right). \quad (55)$$

The lower and upper membership functions are defined in Table 1. Also, through the Taylor series based approach in (Sala and Ario, 2009), the minimum and maximum values of $f_1(\mathbf{x}(t))$ and $f_2(\mathbf{x}(t))$ can be obtained in

Table 1: Lower and Upper Membership Functions for the IT2 Polynomial Fuzzy Model of the Inverted Pendulum.

Lower and upper membership functions	
$\begin{aligned} \underline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) &= \underline{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t))) \\ &= \frac{f_{1\max} - f_1(\mathbf{x}(t))}{f_{1\max} - f_{1\min}}; \end{aligned}$	$\begin{aligned} \underline{\mu}_{\tilde{M}_2^1}(f_2(\mathbf{x}(t))) &= \underline{\mu}_{\tilde{M}_2^3}(f_2(\mathbf{x}(t))) \\ &= \frac{f_{2\max} - f_2(\mathbf{x}(t))}{f_{2\max} - f_{2\min}}; \end{aligned}$
$\begin{aligned} \bar{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) &= \bar{\mu}_{\tilde{M}_1^4}(f_1(\mathbf{x}(t))) \\ &= \frac{f_1(\mathbf{x}(t)) - f_{1\min}}{f_{1\max} - f_{1\min}}; \end{aligned}$	$\begin{aligned} \bar{\mu}_{\tilde{M}_2^2}(f_2(\mathbf{x}(t))) &= \bar{\mu}_{\tilde{M}_2^4}(f_2(\mathbf{x}(t))) \\ &= \frac{f_2(\mathbf{x}(t)) - f_{2\min}}{f_{2\max} - f_{2\min}} \end{aligned}$
with $x_2(t) = 0, m_p = m_{p\max}$	with $m_p = m_{p\max}$
$= 2\text{kg}$ and $M_c = M_{c\min} = 18\text{kg}$	$= 1\text{kg}$ and $M_c = M_{c\max} = 20\text{kg}$
$\begin{aligned} \bar{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) &= \bar{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t))) \\ &= \frac{f_{1\max} - f_1(\mathbf{x}(t))}{f_{1\max} - f_{1\min}}; \end{aligned}$	$\begin{aligned} \bar{\mu}_{\tilde{M}_2^1}(f_2(\mathbf{x}(t))) &= \bar{\mu}_{\tilde{M}_2^3}(f_2(\mathbf{x}(t))) \\ &= \frac{f_{2\max} - f_2(\mathbf{x}(t))}{f_{2\max} - f_{2\min}} \end{aligned}$
$\begin{aligned} \underline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) &= \underline{\mu}_{\tilde{M}_1^4}(f_1(\mathbf{x}(t))) \\ &= \frac{f_1(\mathbf{x}(t)) - f_{1\min}}{f_{1\max} - f_{1\min}}; \end{aligned}$	$\begin{aligned} \underline{\mu}_{\tilde{M}_2^2}(f_2(\mathbf{x}(t))) &= \underline{\mu}_{\tilde{M}_2^4}(f_2(\mathbf{x}(t))) \\ &= \frac{f_2(\mathbf{x}(t)) - f_{2\min}}{f_{2\max} - f_{2\min}}; \end{aligned}$
with $x_2(t) = x_{2\max}, m_p = m_{p\max}$	with $m_p = m_{p\min} = 1\text{kg}$
$= 2\text{kg}$ and $M_c = M_{c\min} = 18\text{kg}$	and $M_c = M_{c\min} = 18\text{kg}$

polynomial functions as:

$$\begin{aligned} f_{1\min}(\mathbf{x}(t)) &= 0.4794x_1(t)^2 + 9.9510, \\ f_{1\max}(\mathbf{x}(t)) &= 0.4794x_1(t)^2 + 15.8915, \\ f_{2\min}(\mathbf{x}(t)) &= 0.0059x_1(t)^2 - 0.0822, \\ f_{2\max}(\mathbf{x}(t)) &= 0.0059x_1(t)^2 - 0.0279. \end{aligned}$$

The lower and upper grades of membership are, respectively, $w_i^L(\mathbf{x}(t)) = \underline{\mu}_{\tilde{M}_1^i}(f_1(\mathbf{x}(t))) \times \underline{\mu}_{\tilde{M}_2^i}(f_2(\mathbf{x}(t)))$ and $w_i^U(\mathbf{x}(t)) = \bar{\mu}_{\tilde{M}_1^i}(f_1(\mathbf{x}(t))) \times \bar{\mu}_{\tilde{M}_2^i}(f_2(\mathbf{x}(t)))$ for all i .

Based the IT2 PFMB fuzzy model, a two-rule IT2 polynomial fuzzy controller is adopted to drive the states of the inverted pendulum to track

those of the reference model.

The following two-rule IT2 polynomial fuzzy controller is adopted to describe the inverted pendulum:

Rule j : IF $x_1(t)$ is \tilde{N}^j

$$\text{THEN } \mathbf{u}(t) = \mathbf{F}_j(\mathbf{x}(t))\hat{\mathbf{e}}(t) + \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t)), \quad j = 1, 2. \quad (56)$$

After combining of all the fuzzy rules, we have

$$\begin{aligned} \mathbf{u}(t) = & \tilde{m}_1(x_1(t))(\mathbf{F}_1(\mathbf{x}(t))\hat{\mathbf{e}}(t) + \mathbf{G}_1(\mathbf{x}(t))\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t))) \\ & + \tilde{m}_2(x_1(t))(\mathbf{F}_2(\mathbf{x}(t))\hat{\mathbf{e}}(t) + \mathbf{G}_2(\mathbf{x}(t))\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t))), \end{aligned} \quad (57)$$

where $\tilde{m}_1(x_1(t))$ and $\tilde{m}_2(x_1(t))$ are the IT2 membership functions of the polynomial fuzzy controller. The upper and lower bounds of the membership functions of the fuzzy controller are defined as follows:

$$\overline{m}_1(x_1(t)) = \begin{cases} 0 & \text{for } x_1(t) < -\frac{5\pi}{12} \\ \frac{x_1(t)+5\pi/12}{5\pi/12} & \text{for } -\frac{5\pi}{12} \leq x_1(t) \leq 0 \\ \frac{5\pi/12-x_1(t)}{5\pi/12} & \text{for } 0 \leq x_1(t) \leq \frac{5\pi}{12} \\ 0 & \text{for } x_1(t) > \frac{5\pi}{12} \end{cases} \quad (58)$$

$$\underline{m}_1(x_1(t)) = \begin{cases} 0 & \text{for } x_1(t) < -\frac{5\pi}{12} \\ \frac{0.9(x_1(t)+5\pi/12)}{5\pi/12} & \text{for } -\frac{5\pi}{12} \leq x_1(t) \leq 0 \\ \frac{0.9(5\pi/12-x_1(t))}{5\pi/12} & \text{for } 0 \leq x_1(t) \leq \frac{5\pi}{12} \\ 0 & \text{for } x_1(t) > \frac{5\pi}{12} \end{cases} \quad (59)$$

$\overline{m}_2(x_1(t)) = 1 - \underline{m}_1(x_1(t))$, $\underline{m}_2(x_1(t)) = 1 - \overline{m}_1(x_1(t))$, and $\tilde{m}_2(x_1(t)) = 1 - \tilde{m}_1(x_1(t))$. The type reductions for the controller are chosen as $\underline{\kappa}_j(x_1(t)) = \overline{\kappa}_j(x_1(t)) = 0.5$, $j = 1, 2$.

The reference model is chosen as $\mathbf{A}_r = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$, $\mathbf{B}_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $r(t) = 2 \sin(0.3t)$.

By dividing the operating domain of $x_1(t)$ into 10 uniform sub-domains (i.e., $L = 10$), we have $\underline{x}_{1l} = -\frac{6\pi}{12} + \frac{\pi}{12}l$ and $\bar{x}_{1l} = -\frac{5\pi}{12} + \frac{\pi}{12}l$, $l = 1, 2, \dots, 10$, which are the lower and upper bounds of the l -th sub-domains of the operating domain, respectively. It is chosen that $\mathbf{D} = \text{diag}\{1, 0\}$.

By applying the stability conditions in Theorem 2, it is chosen that $m_p = 1\text{kg}$, $M_c = 19\text{kg}$, $\varepsilon_1(\mathbf{x}) = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6(\mathbf{x}, \mathbf{x}_r) = \varepsilon_7(\mathbf{x}, \mathbf{x}_r) = \varepsilon_8(\mathbf{x}, \mathbf{x}_r) = 0.001$. $\mathbf{F}_j(x_1(t))$ and $\mathbf{G}_j(x_1(t))$, $j = 1, 2$ are polynomials with monomials in x_1 of degree 2.

After the stability conditions in the form of SOS have been solved, the polynomial feedback gains are obtained as $\mathbf{F}_1 = [151.7754x_1^2 - 16.2622x_1 + 2016.1233 \quad 58.9663x_1^2 - 7.9721x_1 + 768.8719]$, $\mathbf{F}_2 = [75.6951x_1^2 + 3.5550x_1 + 1750.8212 \quad 29.5860x_1^2 - 0.1111x_1 + 661.3961]$, $\mathbf{G}_1 = [-0.7514x_1^2 + 1.0943x_1 + 325.7521 \quad -3.0519x_1^2 + 0.3474x_1 + 78.7202]$, $\mathbf{G}_2 = [3.7764x_1^2 + 2.0589x_1 + 302.9341 \quad -0.3117x_1^2 + 0.5980x_1 + 71.4883]$ and $\mathbf{X} = \begin{bmatrix} 1.9427 & -4.7345 \\ -4.7345 & 24.4871 \end{bmatrix}$. The minimum values of σ_1 , σ_2 and σ_3 are obtained as 0.7398, 0.0573 and 1.2737, respectively.

By applying the IT2 polynomial fuzzy controller for tracking control with the initial conditions $\mathbf{x}(0) = [\frac{\pi}{12} \quad 0]$ and $\mathbf{x}_r(0) = [\frac{\pi}{16} \quad 0]$, the simulation results of state response and control signal are shown in Figs. 4 to 6, which demonstrate that the tracking errors are sufficiently small.

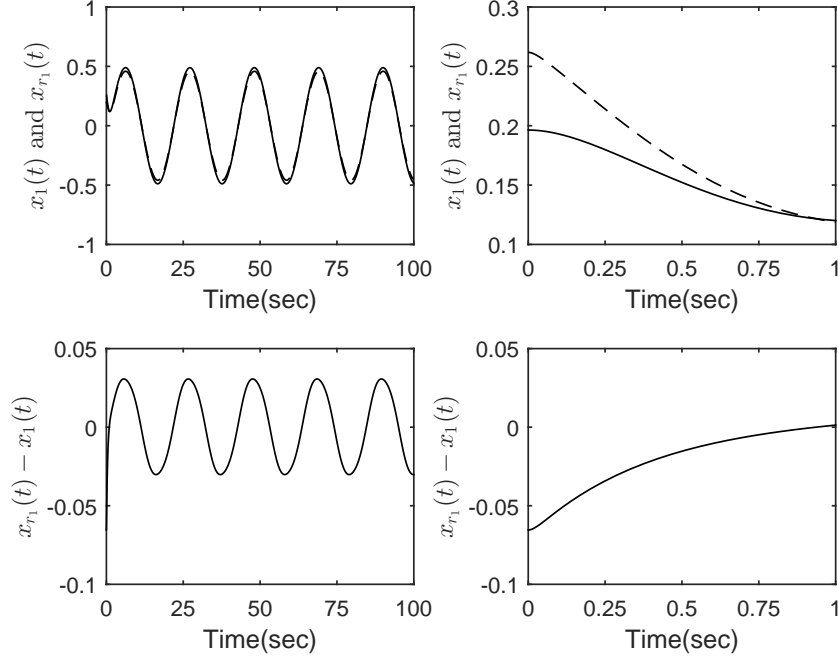


Figure 4: Tracking control performance of the inverted pendulum for $x_1(t)$ under 10 sub-domains approach in the example, $\sigma_1 = 0.7398$ $\sigma_2 = 0.0573$ and $\sigma_3 = 1.2737$. Top left panel: responses of $x_1(t)$ (Dash curve) and $x_{r_1}(t)$ (Solid curve) from 0 to 100 seconds. Top right panel: responses of $x_1(t)$ (Dash curve) and $x_{r_1}(t)$ (Solid curve) from 0 to 1 second. Bottom left panel: response of $x_{r_1}(t) - x_1(t)$ from 0 to 100 seconds. Bottom right panel: response of $x_{r_1}(t) - x_1(t)$ from 0 to 1 second.

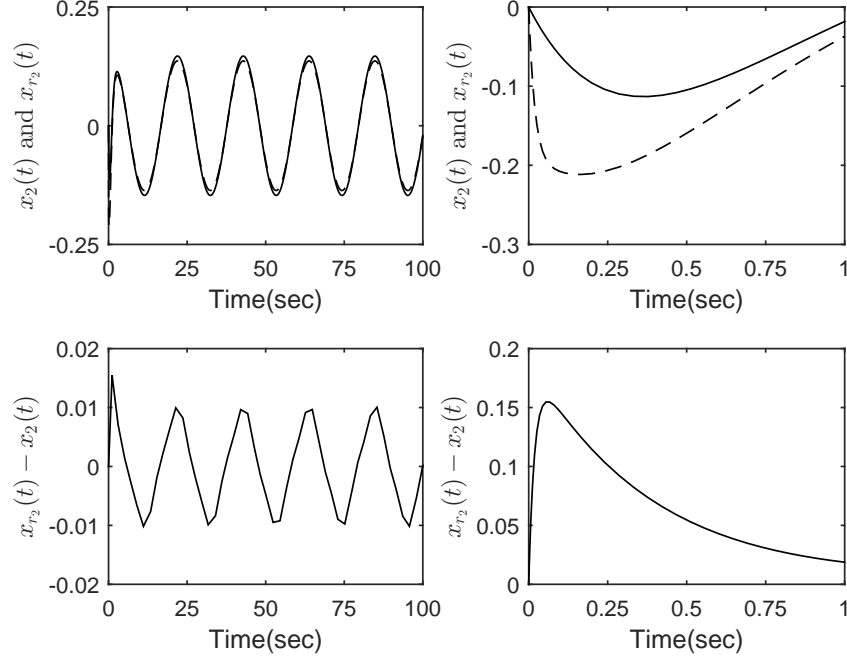


Figure 5: Tracking control performance of the inverted pendulum for $x_2(t)$ under 10 sub-domains approach in the example, $\sigma_1 = 0.7398$ $\sigma_2 = 0.0573$ and $\sigma_3 = 1.2737$. Top left panel: responses of $x_2(t)$ (Dash curve) and $x_{r_2}(t)$ (Solid curve) from 0 to 100 seconds. Top right panel: responses of $x_2(t)$ (Dash curve) and $x_{r_2}(t)$ (Solid curve) from 0 to 1 second. Bottom left panel: response of $x_{r_2}(t) - x_2(t)$ from 0 to 100 seconds. Bottom right panel: response of $x_{r_2}(t) - x_2(t)$ from 0 to 1 second.

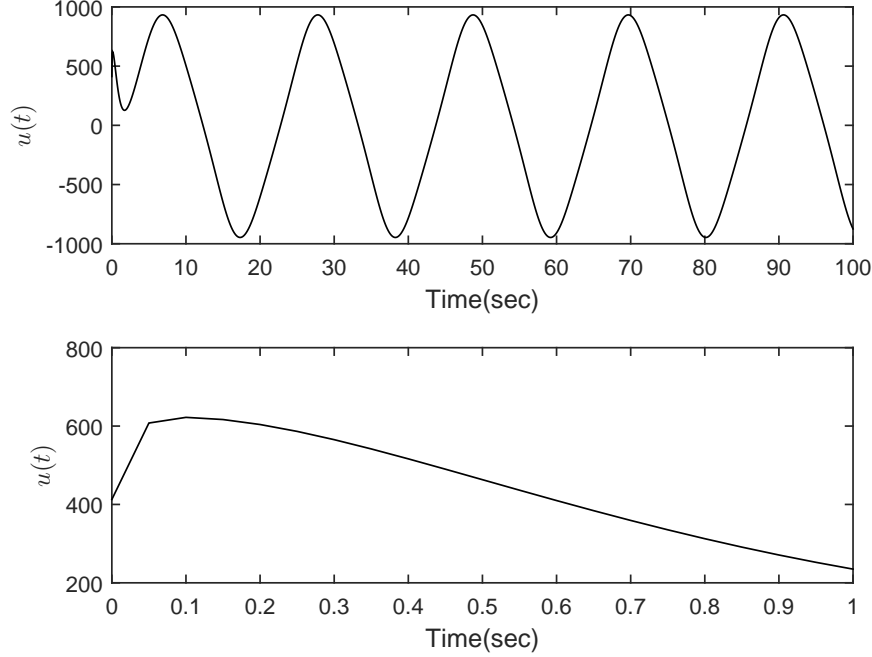


Figure 6: The control signal $u(t)$ under 10 sub-domains approach in the example. Top panel: $u(t)$ from 0 to 100 seconds. Bottom panel: $u(t)$ from 0 to 1 seconds.

Remark 8. When the feedback gains $\mathbf{F}_j(x_1(t))$ and $\mathbf{G}_j(x_1(t))$, $j = 1, 2$, are of degree 0 in $x_1(t)$, with all of other settings being the same, no feasible solution can be found, which verified again the advantage of the polynomial fuzzy controller.

Remark 9. To compare the MFD and MFI approaches, Theorem 1 is applied with the same IT2 polynomial fuzzy model of the inverted pendulum, reference model and all other parameters. Same with the first example, no feasible solution is found, which demonstrates again that introducing the information of IT2 membership functions helps reduce the conservativeness of stability conditions.

5. Conclusion

In this paper, the IT2 PFMB tracking control system with the time-varying delay has been investigated through the Lyapunov-krasovskii functional based approach and the stability conditions are obtained in terms of SOS. The IT2 polynomial fuzzy controller is designed to drive the states of the IT2 fuzzy model to follow those of a stable reference model under time-varying delay situation. The tracking control performance is evaluated by the H_∞ performance index. Furthermore, the MFD approach is adopted to further relax the stability conditions. In the simulation example, the performance of MFD and MFI based approaches are compared meanwhile the IT2 polynomial controller and IT2 constant controller are compared as well. The simulation results demonstrate that the MFD approach help to reduce the conservativeness in the stability conditions.

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